Q 2 The current density in an electron beam is given by $\vec{J} = J_0\left(1-\frac{\rho^2}{b^2}\right)\vec{e}_z$ $(\rho<b)$ where J_0 is a constant an b is the beam radius. At $r=\frac{b}{3}$ the magnetic field is given as $\vec{H} = kbJ_0\,\vec{e}_\phi$. Evaluate k.

A Ampere's law $\oint \vec{H}\cdot\vec{dl} = I_{enc}$

At $r=\frac{b}{3}$ $I_{enc} = \oint \vec{J}\cdot\vec{ds} = \oint\left(J_0\left(1-\frac{\rho^2}{b^2}\right)\vec{e}_z\right)\cdot\left(\rho\, d\rho\, d\phi\,\vec{e}_z\right)$

$$I_{enc} = \int_{\rho=0}^{b/3}\int_{\phi=0}^{2\pi} J_0\left(1-\frac{\rho^2}{b^2}\right)\rho\, d\rho\, d\phi = 2\pi J_0\int_{\rho=0}^{b/3}\left(\rho-\frac{\rho^3}{b^2}\right)d\rho$$

$$I_{enc} = 2\pi J_0\left[\frac{\rho^2}{2}-\frac{\rho^4}{4b^2}\right]\Bigg|_{\rho=0}^{b/3} = 2\pi J_0\left(\frac{b^2}{18}-\frac{b^2}{324}\right)$$

$$I_{enc} = \frac{17\pi J_0 b^2}{162}$$

$$\oint \vec{H}\cdot\vec{dl} = \int_{\phi=0}^{2\pi}(H_\phi\vec{e}_\phi)\cdot(\rho\, d\phi\,\vec{e}_\phi) = 2\pi\rho\, H_\phi\Big|_{\rho=b/3} = \frac{2\pi b H_\phi}{3}$$

$$\frac{2\pi b H_\phi}{3} = \frac{17\pi b^2 J_0}{162} \Rightarrow H_\phi = \frac{17 b J_0}{108}$$

$$\vec{H} = H_\phi\vec{e}_\phi = \frac{17bJ_0}{108}\vec{e}_\phi$$

$$\frac{17\cancel{b}\cancel{J_0}}{108}\vec{e}_\phi = k\cancel{b}\cancel{J_0}\,\vec{e}_\phi \Rightarrow k = \frac{17}{108}$$