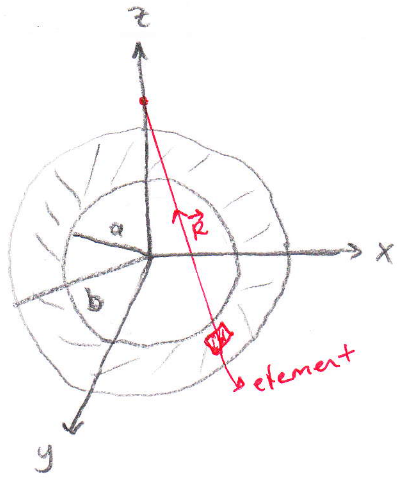
Q Suppose there is a ring shaped conductor which charges are distributed uniformly with a surface density ρ_s [C/m²]. Calculate the electric field on the z-axis.

A



The conductor is at $z=0$

- Electric field can be determined by integrating the contribution of each element of charge distribution.

$$\boxed{\vec{E} = \frac{1}{4\pi\varepsilon}\int_S \frac{dp}{|\vec{R}|^2}\,\vec{e_R}}$$

$$\vec{R} = -\rho\,\vec{e_\rho} + z\,\vec{e_z} \qquad |\vec{R}| = \sqrt{\rho^2+z^2}$$

$$\vec{e_R} = \frac{\vec{R}}{|\vec{R}|} = \frac{-\rho}{\sqrt{\rho^2+z^2}}\,\vec{e_\rho} + \frac{z}{\sqrt{\rho^2+z^2}}\,\vec{e_z}$$

$$dp = \rho_s\,ds = \rho_s\,\rho\,d\rho\,d\phi$$

↳ charge of an element

$$\vec{E} = \frac{1}{4\pi\varepsilon}\int_S \frac{\rho_s\,\rho\,d\rho\,d\phi}{\rho^2+z^2}\left(\frac{-\rho}{\sqrt{\rho^2+z^2}}\,\vec{e_\rho} + \frac{z}{\sqrt{\rho^2+z^2}}\,\vec{e_z}\right)$$

$$\vec{E} = \frac{\rho_s}{4\pi\varepsilon}\left[\underbrace{\int_S \frac{-\rho^2\,d\rho\,d\phi}{(\rho^2+z^2)^{3/2}}\,\vec{e_\rho}}_{A} + \underbrace{\int_S \frac{\rho z\,d\rho\,d\phi}{(\rho^2+z^2)^{3/2}}\,\vec{e_z}}_{B}\right]$$

$$A = \int_S \frac{-\rho^2 \, d\rho \, d\phi}{(\rho^2+z^2)^{3/2}} \vec{e_\rho} \quad , \quad \vec{e_\rho} = \cos\phi \, \vec{e_x} + \sin\phi \, \vec{e_y}$$

$$A = \int_{\rho=a}^{b} \int_{\phi=0}^{2\pi} \frac{-\rho^2 \, d\rho \, d\phi}{(\rho^2+z^2)^{3/2}} (\cos\phi \, \vec{e_x} + \sin\phi \, \vec{e_y})$$

$$A = \int_{\rho=a}^{b} \int_{\phi=0}^{2\pi} \frac{-\rho^2 \cos\phi \, d\rho \, d\phi}{(\rho^2+z^2)^{3/2}} \vec{e_x} + \int_{\rho=a}^{b} \int_{\phi=0}^{2\pi} \frac{-\rho^2 \sin\phi \, d\rho \, d\phi}{(\rho^2+z^2)^{3/2}} \vec{e_y}$$

Integrating $\cos\phi$ and $\sin\phi$ from 0 to 2π gives 0.

$$A = 0$$

$$B = \int_S \frac{\rho z \, d\rho \, d\phi}{(\rho^2+z^2)^{3/2}} \vec{e_z} \quad \Rightarrow \quad \begin{aligned} \rho^2 + z^2 &= u \\ 2\rho \, d\rho &= du \end{aligned}$$

$$B = z \int_{\rho=a}^{b} \frac{du}{2u^{3/2}} \underbrace{\int_{\phi=0}^{2\pi} d\phi}_{2\pi} \vec{e_z} = (2\pi z)\left(-\frac{1}{\sqrt{u}}\right) = \left.\frac{-2\pi z}{\sqrt{\rho^2+z^2}}\right|_{\rho=a}^{b} \vec{e_z}$$

$$B = -2\pi z \left(\frac{1}{\sqrt{b^2+z^2}} - \frac{1}{\sqrt{a^2+z^2}} \right) \vec{e_z}$$

$$\vec{E} = \frac{\rho_s}{4\pi\varepsilon} \cdot B = \frac{\rho_s 2\pi z}{4\pi\varepsilon} \left(\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}} \right) \vec{e_z}$$

$$\boxed{\vec{E} = \frac{\rho_s z}{2\varepsilon} \left(\frac{1}{\sqrt{a^2+z^2}} - \frac{1}{\sqrt{b^2+z^2}} \right) \vec{e_z}}$$