Q Determine the electric field intensity both inside and outside a spherical cloud of electrons with a uniform volume charge density

$$\rho_v = \begin{cases} -\rho_o \ (\rho_o : \text{positive}), & 0 \le r \le b \\ 0, & r > b \end{cases}$$

by solving Laplace's and Poisson's equations for V.

A In spherical coordinates

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr} V\right) + \underbrace{\frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d}{d\theta} V\right)}_{0} + \underbrace{\frac{1}{r^2 \sin^2\theta}\frac{d^2}{d\phi^2} V}_{0}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr} V\right)$$

Inside the cloud

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} = \frac{\rho_o}{\varepsilon}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr} V\right) = \frac{\rho_o}{\varepsilon} \Rightarrow \frac{d}{dr}\left(r^2 \frac{d}{dr} V\right) = \frac{\rho_o r^2}{\varepsilon}$$

$\Leftarrow$ by integrating

$$r^2 \frac{d}{dr} V = \frac{\rho_o r^3}{3\varepsilon} + C_1 \Rightarrow \boxed{\frac{d}{dr} V = \frac{\rho_o r}{3\varepsilon} + \frac{C_1}{r^2}}$$

We know that $\vec{E} = -\nabla V = -\frac{d}{dr} V \vec{e}_r = \left[-\frac{\rho_o r}{3\varepsilon} - \frac{C_1}{r^2}\right]\vec{e}_r$

- E cannot be infinite at $r = 0$ then C_1 must be vanish

$$\boxed{\vec{E}_{inside} = \frac{-\rho_o r}{3\varepsilon}\vec{e}_r \qquad 0 \le r \le b}$$

Outside the cloud

$\nabla^2 V = 0$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0 \quad \Rightarrow \quad \frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0$$

by integrating

$$r^2\frac{dV}{dr} = C_2 \quad \Longrightarrow \quad \boxed{\frac{dV}{dr} = \frac{C_2}{r^2}}$$

$$\text{Again} \quad \vec{E} = -\nabla V = -\frac{d}{dr}V\,\vec{e_r} = -\frac{C_2}{r^2}\vec{e_r}$$

$$\vec{E}_{outside} = -\frac{C_2}{r^2}\vec{e_r}$$

Boundary Conditions

$$\text{At} \quad r = b \qquad \vec{E}_{inside} = \vec{E}_{outside}$$

$$-\frac{\rho_0 b}{3\varepsilon} = -\frac{C_2}{b^2} \quad \Longrightarrow \quad C_2 = \frac{\rho_0 b^3}{3\varepsilon}$$

$$\boxed{\vec{E}_{outside} = -\frac{C_2}{r^2}\vec{e_r} = -\frac{\rho_0 b^3}{3\varepsilon r^2}\vec{e_r} \quad r > b}$$